# PHI 303 Problem Set #6

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## Soundness of $\mathcal{ND}_s$

**Definition 1** (Sheffer Stroke)**.** Let '$\uparrow$' denote the binary connective defined by the following truth table:

| $\phi$ | $\psi$ | $\phi$ | $\uparrow$ | $\psi$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | T | T | F |
| F | T | F | T | T |
| F | F | F | T | F |

**Definition 2.** Let $\mathcal{L}_s$ denote the language that consists of sentence letters, $\bot$, and wffs of the form $\phi \uparrow \psi$ (or $\phi|\psi$). That is, every wff in $\mathcal{L}_s$ is formed using at most the set of connectives $\{\uparrow\}$, which we know is expressively adequate.

**Definition 3** ($\mathcal{L}_s$-Structures)**.** An $\mathcal{L}_s$-structure, $\mathcal{M}$, is a (total) function from the set of wffs in $\mathcal{L}_s$ to the set of truth-values $V = \{T, F\}$, i.e. $f_{\mathcal{M}} : \mathcal{L}_s \to V$.

**Definition 4** (Truth Functions in $\mathcal{L}_s$)**.** A truth function $f_n$ is a total function from a(n) (ordered) sequence of $n$ truth-values to the set of truth-values. The following are all truth functions in $\mathcal{L}_s$:

$$\textbf{nand}(x, y) = \begin{cases} F & \text{if } x = y = T \\ T & \text{otherwise} \end{cases}$$

$$\textbf{Falsum}(x, y) = F$$

**Definition 5** (Satisfaction)**.** For any (possibly empty, finite or infinite) set of wffs $\Gamma$, if $[\![\phi]\!]_{\mathcal{M}} = T$ for every $\phi \in \Gamma$, then $\mathcal{M}$ *satisfies* $\Gamma$.

**Definition 6** (Entailment)**.** If every $\mathcal{L}_s$-structure that satisfies $\Gamma$ also satisfies $\phi$, then $\Gamma$ entails $\phi$, i.e. $\Gamma \vDash_s \phi$.

**Definition 7.** Let $\mathcal{ND}_s$ be the the proof system that consists of the following rules:

## Sheffer Stroke

$$
\begin{array}{l|l l}
l & \phi & \\
m & \quad \begin{array}{|l} \psi \\ \hline \bot \end{array} & \\
n & & \\
 & \phi|\psi & |\mathrm{I},\ l,\ m\text{–}n
\end{array}
$$

$$
\begin{array}{l|l l}
l & \phi|\psi & \\
m & \phi & \\
n & \psi & \\
 & \bot & |\mathrm{E},\ l,\ m,\ n
\end{array}
$$

## Contradiction

$$
\begin{array}{l|l l}
m & \bot & \\
 & \phi & \bot\mathrm{E},\ m
\end{array}
$$

**Definition 8** (Step-Soundness)**.** A line of a proof is *step-sound* iff the assumptions on which that line depends entail the sentence on that line. Assumptions themselves are treated as self-dependent.

**Definition 9** (Rule-Soundness)**.** An inference rule, $\mathcal{R}$, is *rule-sound* iff for all $\mathcal{ND}_s$ proofs, if we obtain a line on a $\mathcal{ND}_s$ proof by applying $\mathcal{R}$, and every earlier line in the proof is step-sound, then our new line is also step-sound.

**Lemma 1** (Rule-Soundness of $\mathcal{ND}_s$)**.** *Every inference rule, $\mathcal{R}$, in $\mathcal{ND}_s$ is rule-sound.*

*Proof.* The proof of rule-soundness of $\mathcal{ND}_s$ follows directly from proving rule-soundness of all its inference rules, namely, |I, |E, and ⊥E (Lemmas 2, 3, and 4): ■

*** For the following Lemmas (2, 3, and 4), let '$\Delta_i$' denote the assumptions (if any) on which line $i$ depends in the corresponding $\mathcal{ND}_s$ proof.

**Lemma 2.** *|I is rule-sound.*

*Proof.* Assume that every line before line $k$ on some $\mathcal{ND}_s$ proof is step-sound, and that |I is used on line $k$. So the situation is:

$$
\begin{array}{r|l@{\qquad}l}
h & \phi & \\
i & \quad \begin{array}{|l} \psi \\ \hline \end{array} & \\
j & \quad \begin{array}{|l} \bot \end{array} & \\
k & \phi|\psi & |\mathrm{I},\ h,\ i\text{–}j
\end{array}
$$

Let $\mathcal{M}$ be any $\mathcal{L}_s$ structure that satisfies $\Delta_k$. Note that all of $\Delta_h$ are among $\Delta_k$. By hypothesis, line $i$ and line $j$ are step-sound. However, no structure can satisfy '$\bot$', so no structure can make all of $\Delta_j$ true. And since the wffs $\Delta_i$ are just the wffs $\Delta_j$, no structure can satisfy all of $\Delta_i$ either. Thus, since $\mathcal{M}$ satisfies both $\Delta_h$ and $\Delta_k$, it cannot satisfy $\psi$, and so it must satisfy $\phi|\phi$. Therefore, $\Delta_k \vDash_s \phi|\phi$, and by definition, line $k$ is step-sound. It follows that since every line in the proof is step-sound, |I is rule-sound.

■

**Lemma 3.** *$|E$ is rule-sound.*

*Proof.* Assume that every line before line $k$ on some $\mathcal{ND}_1$ proof is step-sound, and that →E is used on line $k$. So the situation is:

$$
\begin{array}{r|l@{\qquad}l}
h & \phi|\psi & \\
i & \phi & \\
j & \psi & \\
k & \bot & |\mathrm{E},\ h,\ i,\ j
\end{array}
$$

Let $\mathcal{M}$ be any $\mathcal{L}_s$ structure that satisfies $\Delta_k$. Note that all of $\Delta_h$, $\Delta_i$, and $\Delta_j$ are among $\Delta_k$. By hypothesis, line $h$, line $i$, and line $j$ are step-sound. Thus, any structure that satisfies $\Delta_k$ would have to satisfy $\phi|\psi$, $\phi$, and $\psi$. However, since no structure can do that, it follows that no structure can satisfy $\Delta_k$, and so $\Delta_k \vDash_s \bot$, vacuously. Therefore, by definition, line $k$ is step-sound, and it follows that since every line in the proof is step-sound, |E is rule-sound.

■

**Lemma 4.** *$\bot E$ is rule-sound.*

*Proof.* Assume that every line before line $n$ on some $\mathcal{ND}_1$ proof is step-sound, and that ⊥E is used on line $k$. So the situation is:

$$
\begin{array}{r|l@{\qquad}l}
m & \bot & \\
n & \phi & \bot\mathrm{E},\ m
\end{array}
$$

Let $\mathcal{M}$ be any $\mathcal{L}_1$ structure that satisfies $\Delta_n$. Note that all of $\Delta_m$ are among $\Delta_n$. By hypothesis, line $m$ is step-sound, which would mean that any structure that satisfies $\Delta_m$ would satisfy '$\bot$'. However, there is no such structure, as nothing can satisfy '$\bot$', so nothing can satisfy $\Delta_m$. It follows that since $\mathcal{M}$ satisfies $\Delta_n$, it must satisfy any wff, $\phi$. It is also worth noting that "from the false anything follows" (EFQ - The Principle of Explosion), that is, anything can be proven from a contradiction. Conclusively, $\Delta_n \vDash_s \phi$, and by definition, line $n$ is step-sound, so $\bot$E is rule-sound.

■

**Lemma 5** (Step-Soundness of $\mathcal{ND}_s$ Proofs)**.** *Every line of every $\mathcal{ND}_s$ proof is step-sound.*

*Proof.* From the Lemmas above, we may observe that every line of any $\mathcal{ND}_s$ proof is obtained by applying some $\mathcal{ND}_s$ rule, all of which are rule-sound. More formally, suppose we fix any line, $n$, on any $\mathcal{ND}_s$ proof. The wff written on line $n$ must be obtained using a formal inference rule which, given the Lemma of Rule-Soundness of $\mathcal{ND}_s$, is rule-sound. This is to say that, if every earlier line is step-sound, then line $n$ itself will be step-sound. Hence, by (strong) induction on the length of $\mathcal{ND}_s$ proofs, every line of every $\mathcal{ND}_s$ proof is step-sound.

■

*** Having proved step-soundness of $\mathcal{ND}_s$, we now know that we have never gone astray and the proof for soundness of $\mathcal{ND}_s$ follows quite simply.

**Theorem 1** (Soundness of $\mathcal{ND}_s$)**.** *For any set of wffs $\Gamma$ and wff $\phi$ in $\mathcal{L}_s$, $\Gamma \vdash_s \phi \Rightarrow \Gamma \vDash_s \phi$.*

*Proof.* Suppose $\Gamma \vdash_s \phi$. It follows that there is an $\mathcal{ND}_s$ proof with $\phi$ appearing on its last line, whose only undischarged assumptions are among $\Gamma$. Given the Lemma of Step-Soundness of $\mathcal{ND}_s$, every line on every $\mathcal{ND}_s$ proof is step-sound, and so it follows that this last line is step-sound. Thus, $\Gamma \vDash_s \phi$. ■