PHI 303 Problem Set #3

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1 Problems

Lemma 2 (Open Trees have Satisfiable Roots)**.** *If Γ is satisfiable, i.e. $[\![\Gamma]\!]_{\mathcal{M}} = T$ for some structure $\mathcal{M}$, then there is at least one open branch on any tree (completed or uncompleted) whose root is Γ.*

Proof. Suppose that Γ is satisfiable, i.e. $[\![\Gamma]\!]_{\mathcal{M}} = T$ for some structure $\mathcal{M}$. Now let $\mathbf{T}$ be a tree whose root is Γ and let $\mathbb{T}$ be a sequence of trees $\mathbf{T}_1, \ldots, \mathbf{T}_z$ such that $\mathbf{T}_1 = \Gamma, \mathbf{T}_z = \mathbf{T}$ and whose $n+1$st element, $\mathbf{T}_{n+1}$, for each $1 \leq n \leq z$ is obtained from the nth element, $\mathbf{T}_n$ by applying a single tree rule. We must show that each element of $\mathbb{T}$ has at least one open branch, whence it follows that $\mathbf{T}$ itself contains at least one open branch. If all formulas on a branch, $\mathbf{B}$, are satisfied by $\mathcal{M}$, then $\mathbf{B}$ cannot contain contradictory literals, since no structure can satisfy both a formula and its negation. Thus, $\mathbf{B}$ must be open. So, for our proof, it suffices to show that every tree in the sequence contains a branch whose formulas are satisfied by some structure $\mathcal{M}$. We proceed by induction on the elements of $\mathbb{T}$:

Base Case: Consider $\mathbf{T}_2$, the second tree in $\mathbb{T}$ and the one obtained by applying a tree rule to $\mathbf{T}_1 = \Gamma$ whose formulas are satisfied by some structure $\mathcal{M}$. If $\mathbf{T}_2$ were to be obtained by the Closure Rule, then by definition, it would form a branch, $\mathbf{B}$, that contains both a formula and its negation. This would result in a closed branch which cannot happen as this would mean that $\mathcal{M}$ does not also satisfy $\mathbf{T}_1 = \Gamma$ (which we have determined is not the case). Therefore, to avoid contradiction, any tree rule that is used to obtain $\mathbf{T}_2$ from $\mathbf{T}_1 = \Gamma$ cannot be the Closure Rule and must yield an open branch, $\mathbf{B}$. This means that all formulas on $\mathbf{B}$ are also satisfied by $\mathcal{M}$ and that the tree $\mathbf{T}_2$ whose root is $\mathbf{T}_1 = \Gamma$ has at least one open branch.

Inductive step: Suppose that some tree $\mathbf{T}_n$ in $\mathbb{T}$ whose root is also Γ contains at least one branch, $\mathbf{B}$, whose formulas are satisfied by $\mathcal{M}$. Similarly to the Base Case, a tree, $\mathbf{T}_{n+1}$, that is a one-step extension of $\mathbf{T}_n$, must have at least one open branch because the structure $\mathcal{M}$ could not satisfy the closed branch $\mathbf{B}$ that contains both a formula and its negation in $\mathbf{T}_{n+1}$ while still satisfying $\mathbf{T}_n$.

By induction on elements of $\mathbb{T}$, it follows that every tree, $\mathbf{T}$, whose root, Γ, is satisfiable, must contain at least one open branch to ensure that for some structure, $\mathcal{M}$, $[\![\Gamma]\!]_{\mathcal{M}} = T$. ∎

Theorem 1 (Soundness of Semantic Trees)**.** *If a tree whose root is $\Gamma \cup \{\neg\phi\}$ is closed, then $\Gamma \vDash \phi$.*

Proof. Suppose that a tree, $\mathbf{T}$, whose root is $\Gamma \cup \{\neg\phi\}$ is closed. If this is the case, then by definition of Closed Branch & Tree, every branch, $\mathbf{B}$, of $\mathbf{T}$ must be closed and contain both an atom and its negation (application of the Closure Rule). Given this definition, since the negation of an atom, namely $\neg\phi$, is given in the tree's root, it follows that the remainder of the root, namely Γ, must contain and thus, yield ϕ. If this is true, then by Lemma 2, there is no structure that satisfies $\Gamma \cup \{\neg\phi\}$. That is, there is no structure, $\mathcal{M}$, for which $\Gamma \cup \{\neg\phi\}$ valuates to true or simultaneously holds. However, having proved that Γ must contain ϕ for **T** to be closed, it follows that every structure, $\mathcal{M}$, that satisfies Γ, must also satisfy ϕ (notice that if $\{\neg\phi\}$ were to be omitted, this would result in an open branch). Thus, by definition of entailment, $\Gamma \vDash \phi$. ∎

Lemma 3 (Satisfiable Roots have Open Trees)**.** *If there is a completed open tree whose root is* Γ*, then* Γ *is satisfiable, i.e.* $[\![\Gamma]\!]_{\mathcal{M}} = T$ *for some structure* $\mathcal{M}$*.*

Proof. Let Γ be the root of some tree, **T**, and suppose that this tree is completed and open. By definition of Closed Branch & Tree, if this tree is open, then it has at least one branch that is not closed via the Closure Rule (applied on two wffs) and thus, does not contain any contradictory literals. Let **B** denote such branch and let $\mathbb{B}$ be a sequence of elements $1 \leq n \leq z$ that contains every wff in **B**, where 1 is its root, Γ, z is its last wff, and the $n+1$st element is obtained by applying a single tree rule to n and checking off the wff. If z is the last wff on **B** and **B** is open, it follows by definition of Completed Branch & Tree that z must be a literal and have a character length (CL) of either 1 or 2. Additionally, since **B** does not contain contradictory literals and z is thus not obtained via the Closure Rule, then there must be some structure, $\mathcal{M}$, that satisfies z. That is, some structure, $\mathcal{M}$, for which z is true ($[\![z]\!]_{\mathcal{M}} = T$). Moreover, by definition of Completed Tree and the fact that all possible tree rules yield either one or two (stacked) wffs, $Z - 1$ must either also be a literal and have a CL of either 1 or 2 or be a checked complex wff and have a CL of least 3. If it is the former, then it is obtained from the same checked wff, $z - 2$, and is then satisfied by the same structure, $\mathcal{M}$. If the latter holds, then it follows form the fact that it is checked and that a rule that is not the Closure Rule has been applied to it, that it must evaluate to T and satisfied by precisely the same structure, $\mathcal{M}$, as Z (which was obtained form it). As all wffs preceding the last checked (complex) wff (which is either $z - 1$ or $z - 2$) will have a CL greater than 3 and be checked to produce the $n+1$st element of $\mathbb{B}$, it follows that all elements of $\mathbb{B}$ must be satisfied by the same structure, $\mathcal{M}$, and the root Γ being itself an element of $\mathbb{B}$, must also have this property. Therefore, Γ is satisfiable and we can conclude that for any completed open tree, which by definition will have at least one open branch, its root (being part of this branch) will be satisfiable, i.e. $[\![\Gamma]\!]_{\mathcal{M}} = T$ for some structure $\mathcal{M}$. ∎

Theorem 2 (Completeness of Semantic Trees)**.** *If* $\Gamma \vDash \phi$*, then a tree whose root is* $\Gamma \cup \{\neg\phi\}$ *is closed.*

Proof. Suppose that $\Gamma \vDash \phi$ for some set of wffs, Γ, and some atom, ϕ. By definition of entailment, every structure, $\mathcal{M}$, that satisfies Γ, must also satisfy ϕ, and so, ϕ must be contained within Γ. If Γ were to be the root of some tree, **T**, from which ϕ would follow, then by Lemma 2, this would result in an open branch and tree. However, if we let the root of **T** be $\Gamma \cup \{\neg\phi\}$, then this would result in a tree that contains both an atom, ϕ, and its negation, $\neg\phi$. Thus, by application of the Closure Rule and definition of Closed Branch & Tree, such tree would be closed and we may infer that if $\Gamma \vDash \phi$, then a tree whose root is $\Gamma \cup \{\neg\phi\}$ will be closed. ∎

Theorem 3 (Decidability)**.** *Any tree with a finite root is completed after some finite number of applications of tree rules.*

Proof. Let **T** be a tree with a finite root. Suppose for *reductio* that **T** is completed after an infinite number of applications of tree rules on its branches. If this is true, then by König's Lemma, **T** has a nonterminating branch that has nonterminating extensions and is infinitely extendable. It follows then, from Lemma 1, that the character count of the tree's root must be greater than or equal to the number of tree rule applications on each of its branches. However, this yields a contradiction because if the number of tree rule applications is infinite, then the tree root's character count and hence, the root itself, cannot be finite (which it is). Therefore, any tree with a finite root, must be completed after some finite number of applications of tree rules (precisely that is less than or equal to the character count of the tree's root). ■