PHI 303 Problem Set #2

Antonella Basso

February 19, 2020

1 Problems

Theorem 1 (Deduction Theorem)**.** $\Gamma, \phi \vDash \psi$ *iff* $\Gamma \vDash \phi \to \psi$.

Proof. Suppose $\Gamma, \phi \vDash \psi$. Then, by definition of entailment, every $\mathcal{L}_1$-structure, $\mathcal{M}$, that satisfies $\Gamma \cup \{\phi\}$ also satisfies $\{\psi\}$. If this is the case, then by definition of satisfaction, $[\![\gamma]\!]_{\mathcal{M}} = T$ for every sentence $\gamma \in \Gamma$, $[\![\phi]\!]_{\mathcal{M}} = T$, and $[\![\psi]\!]_{\mathcal{M}} = T$. This indicates that there is no structure $\mathcal{M}$ in which Γ is satisfied and in which $[\![\phi]\!]_{\mathcal{M}} = T$ while $[\![\psi]\!]_{\mathcal{M}} = F$. Thus, by definition of $\mathcal{L}_1$-valuation, there is no structure $\mathcal{M}$ in which Γ is satisfied while $[\![\phi \to \psi]\!]_{\mathcal{M}} = F$. That is, if $\mathcal{M}$ satisfies Γ, then it must be the case that $[\![\phi \to \psi]\!]_{\mathcal{M}} = T$, and by definition of satisfaction, it follows that $\mathcal{M}$ satisfies $(\phi \to \psi)$. Therefore, having proved that $\mathcal{M}$ satisfies both Γ and $(\phi \to \psi)$, by definition of entailment, $\Gamma \vDash \phi \to \psi$.

Now suppose $\Gamma \vDash \phi \to \psi$. Then, by definition of entailment, every $\mathcal{L}_1$-structure, $\mathcal{M}$, that satisfies Γ also satisfies $(\phi \to \psi)$. If this is the case, then by definition of satisfaction, $[\![\gamma]\!]_{\mathcal{M}} = T$ for every sentence $\gamma \in \Gamma$ and $[\![\phi \to \psi]\!]_{\mathcal{M}} = T$. This indicates that there is no structure $\mathcal{M}$ in which Γ is satisfied and in which $[\![\psi \to \psi]\!]_{\mathcal{M}} = F$. Thus, by definition of $\mathcal{L}_1$-valuation, there is no structure $\mathcal{M}$ in which Γ is satisfied and $[\![\phi]\!]_{\mathcal{M}} = T$ while $[\![\psi]\!]_{\mathcal{M}} = F$. That is, if $\mathcal{M}$ satisfies Γ and $[\![\phi]\!]_{\mathcal{M}} = T$, then it must be the case that $[\![\psi]\!]_{\mathcal{M}} = T$. Moreover, every $\mathcal{L}_1$-structure, $\mathcal{M}$, that satisfies $\Gamma \cup \{\phi\}$ must also satisfy $\{\psi\}$, and by definition of entailment, it follows that $\Gamma \vDash \phi \to \psi$.

By proving that if $\Gamma, \phi \vDash \psi$ then $\Gamma \vDash \phi \to \psi$ and that if $\Gamma \vDash \phi \to \psi$ then $\Gamma, \phi \vDash \psi$ we conclude that the biconditional holds for any set of wffs, Γ, and wffs ϕ and ψ. ∎

Proposition 1. $\{\uparrow\}$ is expressively adequate.

Proof. (For this you may appeal to the DNF thereom and/or the expressive adequacy of $\{\neg, \vee, \wedge\}$, but *not* the expressive adequacy of $\{\neg, \vee\}$. Use the valuation function of '$\uparrow$' and the standard connectives to prove the requisite equivalencies.)

Let ϕ and ψ be wffs in $\mathcal{L}_1$. Given the truth table for $(\phi \uparrow \psi)$, it is clear that at least one constituent must be false for $(\phi \uparrow \psi)$ to be true. That is, given the valuation function of '$\uparrow$', for some $\mathcal{L}_1$-structure, $\mathcal{M}$, any valuation $[\![\phi \uparrow \psi]\!]_{\mathcal{M}} = T$ iff $[\![\phi]\!]_{\mathcal{M}} = F$ (i.e $[\![\neg\phi]\!]_{\mathcal{M}} = T$) or $[\![\psi]\!]_{\mathcal{M}} = F$ (i.e $[\![\neg\psi]\!]_{\mathcal{M}} = T$). Therefore, the expression $(\phi \uparrow \psi)$ can be expressed by $(\neg\phi \vee \neg\psi)$ and hence, the truth function of '$\uparrow$' can be expressed in terms of *at most* '$\neg$' and '$\vee$' and vise versa. Moreover, given the identical truth tables for both $(\phi \uparrow \psi)$ and $(\neg\phi \vee \neg\psi)$, it follows that every $\mathcal{L}_1$-structure that satisfies $(\phi \uparrow \psi)$, also satisfies $(\neg\phi \vee \neg\psi)$ and vise versa. Then, by definitions of entailment and equivalence, $(\phi \uparrow \psi) \vDash (\neg\phi \vee \neg\psi)$ and $(\neg\phi \vee \neg\psi) \vDash (\phi \uparrow \psi)$, and hence, $(\phi \uparrow \psi)$ and $(\neg\phi \vee \neg\psi)$ are logically equivalent. If the two are logically equivalent, then by definition, for any two wffs ϕ and ψ of the form $(\phi \uparrow \psi)$, there will be a logically equivalent wff of the form $(\neg\phi \vee \neg\psi)$. This being the case, to prove that '$\uparrow$' is expressively adequate, all we need only prove that $\{\neg, \vee\}$ is expressively adequate. To do this, we shall consider the DNF Theorem and the definition of Disjunctive Normal Form, from which it follows that the set of connectives $\{\neg, \vee, \wedge\}$ is expressively adequate on the basis that every truth function can be expressed by a wff in DNF, that is, a wff containing only the connectives '$\neg$', '$\vee$' and '$\wedge$'. Thus, since any wff in DNF contains only these connectives, to prove the expressive adequacy of $\{\neg, \vee\}$, it suffices to show that for any wff in DNF, there is an equivalent wff that only contains *at most* the connectives '$\neg$' and '$\vee$':

Let ϕ and ψ be a wffs in DNF.

Base Case: If $\mathsf{Len}(\phi) = 0$, then ϕ must be an atom and it hence, has no connectives. Therefore ϕ contains *at most* the connectives '$\neg$' and '$\vee$'.

Inductive step: Suppose this is the case for all wffs of length $\leq n$. Then, for some wff ψ with $\mathsf{Len}(\psi) = n+1$, the n+1st connective will either be '$\neg$', '$\vee$', or '$\wedge$'. Since we are trying to prove the expressive adequacy of $\{\neg, \vee\}$, then we need only consider the case in which $\psi = (\gamma \wedge \chi)$. Given that $\gamma \wedge \chi \Leftrightarrow \neg(\neg\gamma \vee \neg\chi)$, it follows that any wff who's main connective is '$\wedge$' can be expressed in terms of '$\neg$' and '$\vee$'. Given this fact and by inductive hypothesis, therefore, every wff in DNF can be expressed *at most* by the connectives '$\neg$' and '$\vee$', and by definition, the set $\{\neg, \vee\}$ is expressively adequate.

Having proved the expressive adequacy of $\{\neg, \vee\}$, it follows that $\{\uparrow\}$ is also expressively adequate in virtue of the logical equivalence of $(\phi \uparrow \psi)$ and $(\neg\phi \vee \neg\psi)$. That is, since every truth function can be expressed by a wff in DNF, every wff in DNF can be expressed in terms of *at most* '$\neg$' and '$\vee$', and every wff with connectives '$\neg$' and '$\vee$' can be expressed by a wff with *at most* the connective '$\uparrow$', then $\{\uparrow\}$ is expressively adequate. ∎

Lemma 1. *If $\phi \Leftrightarrow \psi$, then $\phi[\omega/\chi] \Leftrightarrow \psi[\omega/\chi]$.*

Proof. Suppose $\phi \Leftrightarrow \psi$. Then by definition of logical equivalence, $\phi \vDash \psi$ and $\psi \vDash \phi$. If this is the case, then by the Uniform Substitution Theorem, for any wffs ϕ, ψ, ω and χ, if $\phi \vDash \psi$ then $\phi[\omega/\chi] \vDash \psi[\omega/\chi]$, and if $\psi \vDash \phi$ then $\psi[\omega/\chi] \vDash \phi[\omega/\chi]$. Therefore, since $\phi \vDash \psi$ and $\psi \vDash \phi$, then $\phi[\omega/\chi] \vDash \psi[\omega/\chi]$ and $\psi[\omega/\chi] \vDash \phi[\omega/\chi]$, and by definition of logical equivalence, $\phi[\omega/\chi] \Leftrightarrow \psi[\omega/\chi]$. ∎

Proposition 2. Consider the self-dual set of connectives $\{\rightarrow, \rightarrow^{\star}\}$. Is this set of connectives expressively adequate? Prove your answer.

Proof. Having proved the expressive adequacy of $\{\uparrow\}$, that is, that every truth function can be expressed by a wff containing *at most* the connective '$\uparrow$', we need only show that for any wff composed of *at most* '$\uparrow$', there is an equivalent wff that contains *at most* the connectives '$\rightarrow$' and '$\rightarrow^{\star}$' to prove the expressive adequacy of $\{\rightarrow, \rightarrow^{\star}\}$. We can infer from having proved the expressive adequacy $\{\uparrow\}$ by appeal to the DNF Theorem that every truth function that can be expressed by a wff containing *at most* the connective '$\uparrow$' can also be expressed by a wff in DNF. Given that for some wffs ϕ and ψ, $\phi \rightarrow \psi \Leftrightarrow ((\phi \wedge \psi) \vee (\neg\phi \wedge \psi)) \vee (\neg\phi \wedge \neg\psi)$, it follows that the truth function of $\phi \rightarrow \psi$ can be expressed in terms of $\{\uparrow\}$ since its logical equivalent is in DNF. It follows that any wff expressed by *at most* '$\uparrow$', can be expressed by *at most* the connective '$\rightarrow$' and therefore can be expressed by *at most* the connectives '$\rightarrow$' and '$\rightarrow^{\star}$'. This indicates that any truth function can be expressed by a wff containing *at most* the connectives '$\rightarrow$' and '$\rightarrow^\star$' and thus, the self-dual set of connectives $\{\rightarrow, \rightarrow^\star\}$ is expressively adequate.

■

Proposition 3. Find the simplest interpolant for the following entailment:

$$(A \wedge (\neg B \vee C)) \wedge (\neg C \wedge D) \vDash (B \vee D) \wedge (B \rightarrow E)$$

Proof. Let ϕ denote the wff on the left side of the entailment and ψ denote the wff on the right side. Given the Interpolation Theorem which proves that there must be some interpolant I who's atomic wffs are all those that occur in both ϕ and ψ, we can easily find such I by applying to ϕ the algorithmic rules provided for this purpose. Such rules require that we rewrite ϕ (which we shall call the input wff) twice; once by substituting the first atom that appears in the input wff, but that does not appear in ψ, in this case A, by a tautology $\top$ (namely $(A \rightarrow A)$), and once by substituting the same atom by a contradiction $\bot$ (namely $(A \wedge \neg A)$). In what follows, the algorithm requires that we join such two wffs by a disjunction, that we let this newly formed wff be our new input wff, and that we repeat this process with all remaining atoms in ϕ that do not occur in ψ. In this case, we are only left with one such atom C, which after replacing accordingly, gives us our final input wff and interpolant I. However, this results in a relatively complex wff which is not in fact the simplest interpolant of the entailment. To acquire such I, we shall simplify the one obtained by considering what the wff is expressing. Precisely, I expresses that either (ϕ where A is replaced by $\top$ and then C is replaced $\top$) or (ϕ where A is replaced by $\bot$ and then C is replaced $\top$) or (ϕ where A is replaced by $\top$ and then C is replaced $\bot$) or (ϕ where A is replaced by $\bot$ and then C is replaced $\bot$). Semantically and by uniform substitution, such expression translates to $(\phi[\top/A][\top/C] \vee \phi[\bot/A][\top/C] \vee \phi[\top/A][\bot/C] \vee \phi[\bot/A][\bot/C])$, and yields the simplest interpolant I for the entailment above. ■