# PHI 303 Problem Set #1

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## 1 Problems

**Proposition 1.** For any sets, $A, B$, $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

*Proof.* Suppose $A$ is a subset of $B$ ($A \subseteq B$), and let $x$ be an arbitrary element in $\mathcal{P}(A)$ ($x \in \mathcal{P}(A)$). If $x \in \mathcal{P}(A)$, then by definition of power sets, $x$ is a subset of $A$ ($x \subseteq A$). Moreover, given that $A$ is a subset of $B$ ($A \subseteq B$), it follows that $x$ is also a subset of $B$ ($x \subseteq B$). Having proved that $x \subseteq B$, then by definition of power sets, $x$ is an element of $\subseteq \mathcal{P}(B)$ ($x \in \mathcal{P}(B)$). This proves that if $x \in \mathcal{P}(A)$, then $x \in \mathcal{P}(B)$, that is, by definition of subsets, $\mathcal{P}(A)$ is a subset of $\mathcal{P}(B)$ ($\mathcal{P}(A) \subseteq \mathcal{P}(B)$).

Suppose $\mathcal{P}(A)$ is a subset of $\mathcal{P}(B)$ ($\mathcal{P}(A) \subseteq \mathcal{P}(B)$), and let $x$ be an arbitrary element in $A$ ($x \in A$). If $x \in A$, then $x$ is a subset of $A$ ($x \subseteq A$), and by definition of power sets, is also an element of $\mathcal{P}(A)$ ($x \in \mathcal{P}(A)$). Moreover, given that $\mathcal{P}(A)$ is a subset of $\mathcal{P}(B)$ ($\mathcal{P}(A) \subseteq \mathcal{P}(B)$), by definition of subsets, it follows that $x$ is also an element of $\mathcal{P}(B)$ ($x \in \mathcal{P}(B)$), and therefore a subset of $B$ ($x \subseteq B$), by definition of power sets. If $x \subseteq B$, then by definition of subsets $x \in B$. This proves that if $x \in A$, then $x \in B$, that is, by definition of subsets, $A$ is a subset of $B$ ($A \subseteq B$).

By proving $\mathcal{P}(A)$ is a subset of $\mathcal{P}(B)$ ($\mathcal{P}(A) \subseteq \mathcal{P}(B)$), given that $A$ is a subset of $B$ ($A \subseteq B$), and vise versa, we prove that the biconditional holds for any sets, $A, B$. ■

**Proposition 2.** For any sets, $A, B, C$, if $A \times B \subseteq A \times C$ and $A \neq \emptyset$, then $B \subseteq C$.

*Proof.* Suppose $A \times B \subseteq A \times C$ and $A \neq \emptyset$, and let $< x, y >$ be an arbitrary ordered pair. If $< x, y >$ is an element of the set created by the Cartesian Product of $A$ and $B$ ($< x, y > \in A \times B$), then by definition of Cartesian Product, $x$ is an element of $A$ ($x \in A$) and $y$ is an element of $B$ ($y \in B$). Moreover, given that $A \times B$ is a subset of $A \times C$, $< x, y >$ is also an element of $A \times C$ ($< x, y > \in A \times C$), by definition of subsets. It then follows, by definition of Cartesian Product, that $x$ is an element of $A$ ($x \in A$) and $y$ is an element of $C$ ($y \in C$). Having proved that $y$ is both an element of $B$ ($y \in B$) and an element of $C$ ($y \in C$), then by definition of subsets, $B$ is a subset of $C$ ($B \subseteq C$.)

By proving $B$ is a subset of $C$ ($B \subseteq C$), given that $A \times B \subseteq A \times C$, where $A \neq \emptyset$, we prove that the conditional holds for any sets, $A, B, C$. ■

**Proposition 3.** For any set, $A$, if $A$ has $n$ elements, then $\mathcal{P}$ has $2^n$ elements. (Use mathematical induction, please.)

*Proof.* Let $A_n$ denote a set containing $n$ elements, and $\mathcal{P}(A_n)$ denote the power set of $A_n$.

**Base Case:** The minimum amount of elements a nonempty set can have is 1, so in the case where $n = 1$, the power set of $A_1$, $\mathcal{P}(A_1)$, which consists of all subsets of $A_1$, has a total of $2^n = 2^1 = 2$ elements, that is, itself and the empty set.

**Inductive Step:** Assume for induction that $A_k$ has $n = k$ elements and that $\mathcal{P}(A_k)$ has $2^n = 2^k$ elements. In the case where $n = k + 1$, the set $A_{k+1}$ is made up of the union between set $A_k$ and another element, namely, $k + 1$ ($A_{k+1} = A_k \cup \{k + 1\}$). It follows that every subset of $A_k$ is a also a subset of $A_{k+1}$, so for any subset of $A_{k+1}$, it either contains the element $k + 1$ or it doesn't. In the case that it doesn't, then it is also a subset of $A_k$, and there are $2^k$ of those subsets, according to the inductive hypothesis. In the case that a subset of $A_{k+1}$ does contain the element $k + 1$, it is formed by including it in a subset of $A_k$, and since there are $2^k$ subsets not already containing it, then there are $2^k$ additional subsets that do contain element $k + 1$. Therefore, the total number of subsets in the power set of $A_{k+1}$, $\mathcal{P}(A_{k+1})$, is $2^k + 2^k = 2 \times 2^k = 2^{k+1}$. ■

**Proposition 4.** Let $R$ be a partial order of set $A$ and let $R'$ be the relation in $A$ defined as follows: $x, y \in A$, $xR'y$ if and only if $xRy$ and $x \neq y$. Show that $R'$ is a strict order.

*Proof.* By definition, if $R$ is a partial order of set $A$, then it is transitive, and by the given definition of $R'$, if $R$ is transitive, then $R'$ is also transitive. Moreover, by definition of $R'$, for $xR'y$, it must be the case that $xRy$ when $x \neq y$, that is, it cannot be the case that $xRx$ or $xR'x$, therefore $R'$ is also irreflexive. Further, if $R$ is a partial order, and hence reflexive and anti-symmetric by definition, then every case of $xRy$ and $yRx$ is a case in which $x = y$. However, since by definition of $R'$, for $xR'y$, it must be the case that $xRy$ when $x \neq y$, by modus tollens, it cannot be the case that $xRy$ and $yRx$ such that $x \neq y$, and therefore it cannot be the case that $xR'y$ and $yR'x$, that is, $R'$ must be asymmetric. Having proved that $R'$ on $A$ is transitive, irreflexive, and asymmetric, given that $R$ is a partial order and the properties of $R'$, by definition, it follows that $R'$ is a strict order of $A$. ■

**Proposition 5.** If $f : X \mapsto Y$ is an injection but not a bijection, then $f^{-1}$ is a partial function.

*Proof.* Suppose that the function which maps elements in $X$ to elements in $Y$ ($f : X \mapsto Y$) is injective. Then, by definition of injection, for each element in $Y$, there is at most one element in $X$. This being the case, the number of elements in $Y$ is not restricted by the number of elements in $X$ and there may be elements in $Y$ that do not have a corresponding element in $X$. Since the inverse of $f$ is a function which maps elements in $Y$ to elements in $X$ ($f^{-1} : Y \mapsto X$), the set $Y$ becomes the new domain and $X$ becomes the codomain. Having shown that the number of elements in $Y$ is undefined, it follows that $f^{-1}$ assigns at most one element of $X$ to every element in $Y$, and by definition, $f^{-1}$ is a partial function. ■

**Definition 1.** Let $\mathbb{T}$ be the following sequence:

$$A_1, A_2 \to A_1, (A_3 \to (A_2 \to A_1)), (A_4 \to (A_3 \to (A_2 \to A_1))), \ldots$$

**Proposition 6.** Show, by induction, that every element of $\mathbb{T}$ is true on any valuation on which $A_1$ is true.

*Proof.* Let $\mathbb{T}_n$ be the $n^(th)$ element of $\mathbb{T}$.

**Base Case:** For the first element of the sequence, $\mathbb{T}_1 = A_1$, $A_1$ is true on any valuation on which $A_1$ is true.

**Inductive Step:** Assume for induction that $\mathbb{T}_n$ is true for $n = k$ on any valuation on which $A_1$ is true, that is, $\mathbb{T}_k$ is true. Considering the truth table for a conditional statement, a conditional is true on any valuation on which the consequent is true. For this reason, if the $\mathbb{T}_(k+1)^th$ element takes the form $A_(k+1) \to \mathbb{T}_k$, and $\mathbb{T}_k$ is true, then the truth value of $A_(k+1)$ need not be known to prove that $\mathbb{T}_(k+1)$ is true. Therefore, since $A_1$ is true and hence every element's consequent is true, it follows that every element of $\mathbb{T}$ is true on any valuation on which $A_1$ is true. ■