Assignment # 6

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Exercise 1 (30)**.** Your phone has a 4-digit security passcode. (I just know, ok?) After a few weeks of use, someone could look at your screen and identify which numbers you type most frequently and can determine the which digits are used in your passcode. With that in mind, which of the following types of passcodes is most secure, and why?

(a) One with four distinct digits.

(b) One with three distinct digits.

(c) One with two distinct digits.

(d) One with one distinct digit.

We are assuming that it can't be determined whether a digit is used more than once, e.g., if your passcode was 6566 a ne'er-do-well could determine you used 5 and 6 but not that it was one 5 and three 6's.

Solution 1. The type of passcode that is most secure will be the type that has the most possible combinations (in this case, permutations since the order matters), as it will be the most difficult or take the most tries to guess when only knowing the different numbers used and the fact that there are 4 total digits. To show which type of passcode is most secure we shall derive the total number of possible permutations of each type of passcode as follows:

(a) One with four distinct digits: $\{a, b, c, d\}$
Given that each digit is different and there are 4 places to take up with no repetitions, the different ways we can arrange 4 distinct digits in a 4-digit passcode.is:

$$4! = 24$$

(b) One with three distinct digits: $\{a, a, b, c\}, \{b, b, a, c\}, \{c, c, a, b\}$
Given that there are 3 distinct digits and 4 places to take up, we know that 1 of them must repeat twice. Since we don't know which one, we must cosider each case. However, as each has the same form, we need only figure out every way we can arrange the digits in one such case and multiply the total by 3, to obtain the number of different ways we can arrange 3 distinct digits in a 4-digit passcode.:

$$\frac{4!}{2!} \times 3 = 12 \times 3 = 36$$

Here, we use the formula for finding the number of ways of arranging n objects (of a multiset), of which p of one type are alike, q of a second type are alike, r of a third type are alike, etc (multiplicites):

$$\frac{n!}{p!q!r!...}$$

(c) One with two distinct digits: $\{a, a, a, b\}, \{a, a, b, b\}, \{a, b, b, b\}$
Given that there are 2 distinct digits and 4 places to take up, we know that either 1 of them will repeat three times, the other will repeat three times, or both will repeat exactly twice. Since we don't know which will be the case, we must cosider each scenario. However, as 2 of these cases have the same form, namely, the case where 1 digit repeats three times, we need only figure out every way we can arrange the digits in one such case and multiply the total by 2. Then, we must add this to the total number of ways we can arrange the digits in the third case, namely, the case where both digits repeat exactly twice:

(1) For the case where 1 digit repeats three times (of which we have 2), using the formula for arranging sets with repeated objects given in part (b):

$$\frac{4!}{3!} \times 2 = 4 \times 2 = 8$$

(2) For the case where both digits repeats exactly twice, using the formula for arranging sets with repeated objects given in part (b):

$$\frac{4!}{2!2!} = 6$$

Computing their sum, we obtain $8 + 6 = 14$ different ways we can arrange 2 distinct digits in a 4-digit passcode.

(d) One with one distinct digit: $\{a, a, a, a\}$
Given that there is only 1 distinct digit that can be used to form a 4-digit passcode (which we have access to), there is only 1 possible way to arrange it.

Having computed the total number of possible ways we can arrange each type of passcode, we now know that the most secure type is one with 3 distinct digits, because it yields the largest number of different ways one can arrange the them, namely, 36.

Exercise 2 (30)**.** The number one *googol*, denoted g, is defined to be $g = 10^{100}$. How many positive integer divisors does g have? Prove your answer in a way that does not involve listing out all the divisors.

Solution 2. To determine the total number of positive integers that evenly divide g, it serves us well to break it down into its prime factors. Notice that $g = 10^{100} = 2^{100} \times 5^{100}$. Thus, we know that every power of 2 from 0 to 100, every power of 5 from 0 to 100, and the product of every other possible power combinations of 2 and 5, will constitue every possible divisor of g. To illustrate this, let A and B be the sets of all powers of 2 and 5 from 0 to 100, as follows:

$$A = \{2^i : 0 \leq i \leq 100\} = \{2^0, 2^1, 2^3, 2^4, ..., 2^{100}\}$$
$$B = \{5^k : 0 \leq k \leq 100\} = \{5^0, 5^1, 5^3, 5^4, ..., 5^{100}\}$$

We may then define the set of all possible (positive integer) divisors of g as follows. Let:

$$X = \{xy : x \in A, y \in B\}$$

Since there are 101 elements in A and 101 elements in B, then X must have $101 \times 101 = 101^2$ elements. Thus, g has 101^2 positive integer divisors.

Exercise 3 (40)**.** Recall that a function $f(x)$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$; and is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Prove or disprove each of the following statements:

(a) Every polynomial (in one variable) is either even or odd.

(b) No polynomial (in one variable) is both even and odd.

Proof. To disprove (a) and (b), we need only provide one counter-example for each statement to show that they do not hold: $\square$

(a) If every polynomial (in one variable) is in fact either even or odd, then it would have to be the case that for $f(x) = 1+x$, either $f(-x) = f(x) = 1+x$, or $f(-x) = -f(x) = -(1+x) = -1-x$. However, substituting $-x$ for x in this function, we get:

$$f(-x) = 1 + (-x) = 1 - x$$

Given the definitions of even and odd above, it is clear that this function is neither even nor odd. Thus, it is not the case that every polynomial (in one variable) is either even or odd. That is, there is at least one polynomial (in one variable) that is both even and odd.

(b) If in fact there is no polynomial (in one variable) that is both even and odd, then it would have to be the case that for $f(x) = 0$, either $f(-x) = f(x) = 0$, $f(-x) = -f(x) = -0 = 0$, or neither, but not both. However, it is evident that in this case, $f(x)$ satisfies the conditions for being both even and odd, since $f(x) = f(-x) = -f(x) = 0$. Therefore, it is not the case that no polynomial (in one variable) is both even and odd. In other words, there is at least one polynomial (in one variable) that is both even and odd.