Assignment # 5

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Exercise 1 (30)**.** Prove or disprove that

$$P \Rightarrow (Q \vee R) \text{ and } (P \Rightarrow Q) \vee (P \Rightarrow R)$$

are logically equivalent.

Proof. To prove these statements are logically equivalent we need ony show that their truth tables are the same:

P	Q	R	P	⇒	(Q	∨	R)	(P	⇒	Q)	∨	(P	⇒	R)
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	T	T	T	T	T	F	F
T	F	T	T	T	F	T	T	T	F	F	T	T	T	T
T	F	F	T	F	F	F	F	T	F	F	F	T	F	F
F	T	T	F	T	T	T	T	F	T	T	T	F	T	T
F	T	F	F	T	T	T	F	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	F	T	F	T	F	T	T
F	F	F	F	T	F	F	F	F	T	F	T	F	T	F

Evidently, the main connectives of these expressions are the same. That is, they yield the equivalent truth values under the same truth conditions of P, Q, and R. Thus, the statements $P \Rightarrow (Q \vee R)$ and $(P \Rightarrow Q) \vee (P \Rightarrow R)$ are logically equivalent. □

Exercise 2 (40)**.** We will say that a set of points A in the plane is called *Agnes* if for all pairs of points $P, Q \in A$ the line segment PQ (that is, the line segment connecting P and Q) is contained in A. Let

$$A = \{(x, y) : x \in \mathbb{R},\ y \in \mathbb{R},\ y < x^2\}$$

Prove or disprove: A is *Agnes*.

Proof. To prove that A is not *Agnes*, we need only find one pair of points $P, Q \in A$ such that the the line segment PQ is not in A. Suppose P is the point $(2, 1)$ on the xy-plane, and Q is the point $(-2, 1)$ respectively. Given the definition of A, P is in A because 2 and 1 are both in $\mathbb{R}$ and $1 < 2^2$. Similarly, Q is in A because -2 and 1 are both in $\mathbb{R}$ and $1 < -2^2$. If A is *Agnes*, then PQ is contained in A. So, if there is at least one point on PQ that is not in A, then A is not *Agnes*.
That said, let

$$PQ = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, y = 1\}$$

From this definition of PQ we may gather that the point which crosses the x-axis, $(0, 1)$, on PQ is not in fact in A. This is so, because $1 \nless 0^2$, despite both 0 and 1 being in $\mathbb{R}$. Therefore, since P and Q in this case are a pair points in A whose line segment PQ is not contained in A, it follows that A is not *Agnes*. $\square$

Exercise 3 (30)**.** For $n \in \mathbb{Z},\ n \geq 2$ define

$$U_n = \{z \in \mathbb{C} : z^n = 1\};$$
$$V_n = \{w \in \mathbb{C} : w^n = 2\}.$$

(a) Prove or disprove: for all $z_1, z_2 \in U_n,\ z_1 z_2 \in U_n$.

(b) Prove or disprove: for all $w_1, w_2 \in V_n,\ w_1 w_2 \in V_n$.

(c) Prove or disprove: for all $z \in U_n,\ w \in V_n,\ zw \in V_n$.

Proof.

(a) A complex number being equal to 1 when raised to a power (integer) of 2 or greater is a necessary condition for being an element in the set U_n. Thus for any arbitrary $z_1, z_2 \in U_n$, their product must also be in the set, since $\sqrt[n]{1} \times \sqrt[n]{1} = 1 \times 1 = 1$. That is, for $z_1, z_2 \in U_n$, $z_1^n = 1$ and $z_2^n = 1$, and it follows that

$$z_1 z_2 = \sqrt[n]{1} \times \sqrt[n]{1} = 1 \times 1 = 1 = z^n$$

for some $z \in \mathbb{C}, z \in U_n$. Therefore, for all $z_1, z_2 \in U_n,\ z_1 z_2 \in U_n$.

(b) Due to the fact that all elements of V_n must be complex numbers equal to 2 when raised to the n^{th} power for $n \in \mathbb{Z},\ n \geq 2$, the product of any two such elements is always in V_n so long as $n = 2$. To prove that this is not the case otherwise consider the following example:

Suppose $n = 3$. Then

$$V_3 = \{w \in \mathbb{C} : w^3 = 2\}$$

So, for some $w_1, w_2 \in V_3$, $w_1^3 = 2$ and $w_2^2 = 2$, and it follows that

$$w_1 w_2 = \sqrt[3]{2} \times \sqrt[3]{2} = \sqrt[3]{2}^2 \neq \sqrt[3]{2} \neq w$$

for some $w \in \mathbb{C},\ w \in V_3$. Thus, it is not the case that for all $w_1, w_2 \in V_n,\ w_1 w_2 \in V_n$.

(c) To be an element of V_n, a complex number must be equal to 2 when raised to the power of n for $n \geq 2$. Thus, since all elements of U_n must be complex and equal to 1 when raised to the power of n for $n \in \mathbb{Z},\ n \geq 2$, the product of any such $z \in U_n$ and $w \in V_n$ will always be equal to 2 and hence, an element of V_n. That is, for $z \in U_n$, $w \in V_n$, $z^n = 1$ and $w^n = 2$, and it follows that

$$zw = \sqrt[n]{1} \times \sqrt[n]{2} = 1 \times \sqrt[n]{2} = \sqrt[n]{2} = w$$

for some $w \in \mathbb{C}, w \in V_n$. Therefore, for all $z \in U_n,\ w \in V_n,\ zw \in V_n$. □