Assignment # 4

Antonella Basso

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Exercise 1 (30)**.** Suppose

$$((P \wedge Q) \vee R) \Rightarrow (R \vee S)$$

is a false statement. Find the truth values of P, Q, R, and S.

Solution 1. Given the truth table for a conditional, a conditional statement is only false when the antecedent is true and the consequent is false. Thus, if $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is a false statement, then $(P \wedge Q) \vee R)$ must be true and $(R \vee S)$ false. For $(R \vee S)$ to be false, both R and S must be false, as disjunctions are only false when both disjuncts are false. Since R must be false, given that it is also a disjunct of the antecedent of the conditional statement which must be true, it follows that the other disjunct, $(P \wedge Q)$, must be true. Being a conjuction, which can only be true when both conjuncts are true, it follows that P and Q must both be true. In this way, we can deduce that if $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false, then the truth values of P, Q, R, and S, can only be; true, true, false, false, respectively.

Exercise 2 (40)**.** We say that a set $U \subseteq \mathbb{R}$ is *open* if for all $x \in U$ there exists an open interval I such that $x \in I$ and $I \subseteq U$. Prove that for all pairs of open sets $U_1, U_2 \subseteq \mathbb{R}$ both $U_1 \cup U_2$ and $U_1 \cap U_2$ are open.

Proof. Suppose U_a and U_b are arbitrary open subsets of $\mathbb{R}$. Then:

Case I: Let $U = U_a \cup U_b$. For some arbitrary element $x \in U$, x must also be an element of at least one of the sets U_a or U_b. Given that both sets are open, it follows that there must be some interval I for which $x \in I$ and $I \subseteq U_a$ and/or $I \subseteq U_b$. Thus, $I \subseteq U$, and since x is an arbitrary element of U, all $x \in U$ have this property, and U is an open set. Moreover, since U is the union of a pair of arbitrary open subsets of $\mathbb{R}$, it follows that for any pair of open sets $U_a, U_b \subseteq \mathbb{R}$, $U_a \cup U_b$ is open.

Case II: Let $U = U_a \cap U_b$. For some arbitrary element $x \in U$, x must be an element of both U_a and U_b. Given that both sets are open, it follows that there must be some interval I for which $x \in I$ and $I \subseteq U_a$ and $I \subseteq U_b$. Thus, $I \subseteq U$, and since x is an arbitrary element of U, all $x \in U$ have this property, and U is an open set. Moreover, since U is the intersection of a pair of arbitrary open subsets of $\mathbb{R}$ (which may or may not be the empty set $\emptyset$), it follows that for any pair of open sets $U_a, U_b \subseteq \mathbb{R}$, $U_a \cap U_b$ is open. $\square$

Exercise 3 (30)**.** We say a that set $K \subseteq \mathbb{R}$ is *closed* if $\overline{K} \subseteq \mathbb{R}$ is open. Prove that for all pairs of closed sets $K_1, K_2 \in \mathbb{R}$ both $K_1 \cup K_2$ and $K_1 \cap K_2$ are closed.

Proof. Suppose K_a and K_b are arbitrary closed subsets of $\mathbb{R}$. Then:

Case I: Let $K = K_a \cup K_b$. By DeMorgan's laws, we have:

$$\overline{K} = \overline{K_a} \cap \overline{K_b}$$

Since K_a and K_b are closed sets, then by definition, $\overline{K_a}$ and $\overline{K_b}$ are open. Given the last proof, which states that the intersection of open sets is open, it follows that $\overline{K}$ is open and thus, $\overline{\overline{K}} = K$ is closed. As K is the union of an arbitrary pair of closed subsets of $\mathbb{R}$, it follows that for any pair of closed sets $K_a, K_b \subseteq \mathbb{R}$, $K_a \cup K_b$ is closed.

Case II: Let $K = K_a \cap K_b$. By DeMorgan's laws, we have:

$$\overline{K} = \overline{K_a} \cup \overline{K_b}$$

Since K_a and K_b are closed sets, then by definition, $\overline{K_a}$ and $\overline{K_b}$ are open. Given the last proof, which states that the union of open sets is open, it follows that $\overline{K}$ is open and thus, $\overline{\overline{K}} = K$ is closed. As K is the intersection of an arbitrary pair of closed subsets of $\mathbb{R}$, it follows that for any pair of closed sets $K_a, K_b \subseteq \mathbb{R}$, $K_a \cap K_b$ is closed. $\square$