Plotting the Future - How Will the Garden Grow?

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Introduction:

The following paper provides a detailed outline of an explorative mathematical modeling project. The aim of this group project was to use growth modeling techniques, specifically that of constrained growth, in conjunction with Python programming language to provide a satisfactory solution to the following problem:

(I) A caretaker is planning a garden with a 40ft by 40ft plot, cleared before seeding. The gardener will plant five types of foliage: coneflowers, hostas, sedums, ferns, and ornamental trees. He/she wants to predict how the garden will grow in order to plan the arrangement. The climate in the area has two annual seasons, rainy and dry, each of which lasts about half of the year (the rainy season is January-June and the dry season is July-December). The soil's carrying capacity is one plant per square foot, and a logistic growth model can be employed for each plant species (not including the trees because they are unaffected by the carrying capacity). Moreover, the different plants thrive in different moisture conditions. The growth rates for each of the four species are given below:

- Coneflowers: 0.02 in rainy season, –0.01 in dry season
- Hostas: –0.01 in rainy season, 0.02 in dry season
- Sedums: –0.01 in rainy season, 0.03 in dry season
- Ferns: 0.02 in rainy season, –0.01 in dry season

Ornamental trees (saplings) will grow in size but not in number. After 4 years they will be large enough to produce shade, which is a function of the tree's canopy (the uppermost branches/leaves that block sunlight). Each tree has effectively zero canopy until it reaches 4 years old, at which point its radius begins to grow at a rate of 0.8 ft/year until age 14 when the tree ceases to grow. The area of sun blockage for each tree can be calculated using the formula for the area of a circle, A=πr2, where r is the radius, and it is assumed that no two trees' canopies overlap. The growth rate of the plants is affected by the amount of sunlight they get, and since both coneflowers and sedums require a lot of sunlight, their growth rates must be adjusted as follows:

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- Coneflowers: If the percentage of the garden's total area that is sunny falls below 70%, the growth rate should be 0.1 lower than otherwise, and if it falls below 50%, the growth rate should be 0.2 lower
- Sedums: If the percentage of the garden's total area that is sunny falls below 50%, the growth rate should be 0.05 lower than otherwise

Given this information, the goal of the project is then to simulate the garden's growth over a period of 20 years to predict the plot's contents provided that:

- The value of Δt is 1 day, where t represents time
- The gardener initially populates the garden with:
  - 12 coneflowers
  - 19 hostas
  - 14 sedums
  - 7 ferns
  - 12 ornamental trees (saplings)

(I)

The overall target here is to use the idea of logistic growth to model the growth of the four plants over time, so as to successfully provide a means to plan the arrangement of the garden. In addition to solving the problem above and using Python to provide a working model for predicting the overall growth of the garden, the following questions were explored to provide a more analytic response:

1. What is the total population of the plants after 20 years?
2. What happens when the simulation parameters are changed?
   a. What would the total population be after 10 years? 40 years?
3. Which has the largest effect: changing the initial population of plant and/or tree types, the amount of shade provided by the trees, or the climate?
   a. POPULATION: How would increasing/decreasing the number of ornamental trees that are planted affect the total population after 20 years?
   b. SHADE: How would increasing/decreasing the size of the canopy of the ornamental trees (changing the radius rate) affect the total population after 20 years? How would increasing/decreasing the number of ornamental trees that are planted affect the total population after 20 years?
   c. CLIMATE: If there was only a dry season what would the garden's population look like and how would it differ from if there was only a rainy season?

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Explanation of the Mathematics:

In order to understand this problem, we broke down the problem into three parts to make the transition into Python easier. We started by figuring out the total population of plants without any seasons or canopies. We knew that the growth rate would be a logistic equation, based on the problem, and that we would need to incorporate the carrying capacity of 1 plant per square foot. Since the garden is 1600 sq ft, we concluded that the carrying capacity for the whole garden was 1600 plants. We used the following equation:

$$\frac{dN}{dT} = r\frac{(K-N)}{K}N$$

where K is the carrying capacity and N is the population to calculate the growth of the plants. We know that the garden can only support a certain number of flowers, so we use this equation to model the population coming to a plateau when it reaches a certain point.

Code:

Now that we have an equation to calculate the growth, we can start coding to incorporate the other parts of the problem. We started with a simple **for** loop that loops N amount of times, where N = total time/Δt. This is the total number of simulations. Inside of the **for** loop, we start an equation that calculates the time t in months. This will help us down the line when we start graphing.

```
import math


for i in range(1, N+1): #runs the for loop N amount of times
  t = i*deltaT #calculates the time t in months
```

The next part of the code is an **if** statement that determines whether or not it's between the 4^{th} year and the 14^{th} year. We are starting to look at the growth of the canopies. If it is the 4^{th} year, that means that the trees are fully grown and will start to grow canopies. If it is between the 4^{th} and the 14^{th} year, then that means that the trees' canopies' radius will grow 0.8 ft(radius_rate) every year. By the end of the 14^{th} year, the trees' canopies will have a radius of 8ft and an area of 16□ ft^2. The canopies will provide shade for the garden. The "covered_area" is calculated inside of the **if** statement, which is the total area of the canopies, and it will update only until the 14^{th}

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year. We are using the area of a circle to calculate the area of the canopies. Outside of the **if** statement is the "percentage" equation, which calculates the percent of the garden that is sunny (not covered by the canopy). This is outside the **if** statement so that the percentage will still be calculated after 14 years. The percentage should be 100% before 4 years and constant after the 14th year.

```
if t > 48 and t <= 168 and t%12==0: #checks if 4 years (48 months) have passed and it's less than 14 years (168 months)
  canopy += radius_rate #incresing the size of the canopy by 0.8
  covered_area = trees*(math.pi*(canopy**2)) #calculating the size of the area covered in shade
percentage = 100 - ((covered_area/area)*100) #calculates the percentage of the garden that is still sunny
```

The next part of the code determines what season it is. There are two seasons: the rainy season and the dry season. The rainy season lasts from January-June, or half the year. In order to determine whether or not it's the rainy season, we check the values of t where the remainder from t/12 is less than or equal to 6 (less than or equal to half the year). If it is the rainy season, then the rates will follow the rainy season rates. Otherwise, it's the dry season and the rates will follow the dry season rates (specified in the introduction).

```
if t%12 <= 6: #checks if it's the rainy season
  cone_r = 0.02
  hosta_r = -0.01
  sedum_r = -0.01
  fern_r = 0.02
else: #dry season
  cone_r = -0.01
  hosta_r = 0.02
  sedum_r = 0.03
  fern_r = -0.01
```

The last part of the code is the calculations of the population growth based on the parameters that we set in the last parts. We coded the logistic equation that we discussed in the previous section by adding the population to the logistic equation and then multiplying the whole equation by Δt.

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```
if percentage < 50: #checks if the canopy covers more than 50% of the garden (less than 50% of the garden is sunny)
  conefls_pop += ((cone_r-.2)*((capacity-conefls_pop)/capacity)*conefls_pop)*deltaT #logistic equation to calculate the population
  hostas_pop += (hosta_r*((capacity-hostas_pop)/capacity)*hostas_pop)*deltaT
  sedums_pop += ((sedum_r-.05)*((capacity-sedums_pop)/capacity)*sedums_pop)*deltaT
  ferns_pop += (fern_r*((capacity-ferns_pop)/capacity)*ferns_pop)*deltaT
  total_plants = conefls_pop + hostas_pop + sedums_pop + ferns_pop #calculates the total number of plants in the garden

elif percentage < 70: #checks if the canopy covers 30-50% of the garden (garden is 50-70% sunny)
  conefls_pop += ((cone_r-.1)*((capacity-conefls_pop)/capacity)*conefls_pop)*deltaT
  hostas_pop += (hosta_r*((capacity-hostas_pop)/capacity)*hostas_pop)*deltaT
  sedums_pop += ((sedum_r)*((capacity-sedums_pop)/capacity)*sedums_pop)*deltaT
  ferns_pop += (fern_r*((capacity-ferns_pop)/capacity)*ferns_pop)*deltaT
  total_plants = conefls_pop + hostas_pop + sedums_pop + ferns_pop

else: #canopy covers less than 30% of the garden (more than 70% of the garden is sunny)
  conefls_pop += ((cone_r)*((capacity-conefls_pop)/capacity)*conefls_pop)*deltaT
  hostas_pop += (hosta_r*((capacity-hostas_pop)/capacity)*hostas_pop)*deltaT
  sedums_pop += ((sedum_r)*((capacity-sedums_pop)/capacity)*sedums_pop)*deltaT
  ferns_pop += (fern_r*((capacity-ferns_pop)/capacity)*ferns_pop)*deltaT
  total_plants = conefls_pop + hostas_pop + sedums_pop + ferns_pop

print(total_plants, t, cone_r, hosta_r, sedum_r, fern_r)
```

The first part checks if the percentage (of the garden that is sunny) is less than 50. If it's less than 50, then the code runs and the rate of the coneflowers is decreased by 0.2 and the rate of the sedums is decreased by 0.05. If the percentage is not less than 50, then the code continues to check if the percentage is less than 70. If it's less than 70, then the code runs and the rate of the coneflowers is decreased by 0.1. If the percentage is neither, then the population is calculated without changes to the rates. The total population is all the populations of the plants added together. The last line is a print statement to see the total population after each month. The total population and the Δt are appended into a list so that we can graph our findings in Jupyter.

Exploratory Questions/Analysis:

In response to the first question (1), the model allowed us to predict that the total population of the plants after 20 years would be roughly 84 plants when rounded up correctly, but 83 when converted to an integer by Python. For the purpose of this exploration and analysis, we shall consider the latter of these approximations. After running the entire code and arriving at 83 plants for the final total population of plants in the garden after 20 years (240 months ≅ 7,200 days), it can be assumed that the entire garden ($1600ft^2$) will be covered by the shade from the 12 fully grown trees' canopies (each measuring 8ft in radius and $\pi 8^2$ in area) at this time. This being the case, the growth rate of the coneflowers will be -0.18 in the rainy season and -0.21 in the dry season, and the growth rate of the sedums will be -0.04 in the rainy season and -0.02 in the dry season. The rates of the hostas and the ferns will stay the same as they are unaffected by shade. It is important to note that for simplicity purposes, each month in the model is estimated to have 30 days (this being the reason there are 7,200 time steps in the code). To visually represent and study the behavior of the garden's population as a function of time, a scatter plot containing the garden's overall population per month (in the span of the 20 years) was made in Python by appending each " t " to a list, each calculated total population to another, creating a data frame

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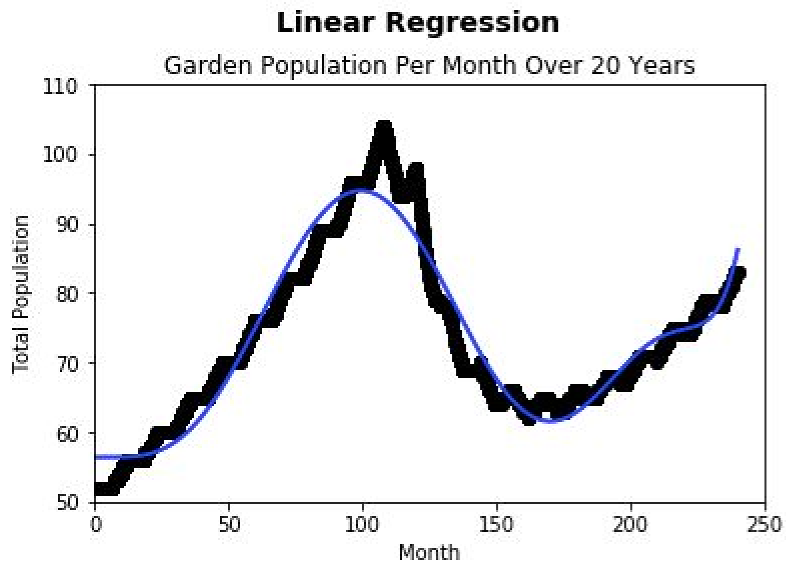
from the two and using the appropriate libraries to map it onto a graph and linear regression. Further, a linear regression model was used on this graph to show the relationship between the garden's population and time. Due to the fact that this relationship is not at all linear, a polynomial of degree 8 was used to make a more accurate approximation. The resulting linear regression graph is shown below:



From this graph, it can be inferred that the rate of the garden's population growth will be largest before it is affected by the shade of the trees. Despite the trees' canopies beginning to grow at month 48, it can be assumed that they would still not be large enough to affect the growth of the plants. At around month 110 however, the graph predicts a rapid decrease in population which can be taken to be a result of the trees' canopies becoming significantly large and beginning to affect growth. Considering the fact that the canopies will cease to grow after the 168^{th} month, it can be observed that although the entire garden will be covered by shade, the garden's population will continue to grow at a positive rate. As shown by the graph, the rate of the garden's population growth after this time will not be as large as the initial growth rate before the presence of shade. This is due to the fact that half of the plant species are affected by shade and will no longer be able to contribute significantly to the growth of the garden population.

The second question (2) can be easily answered by two more linear regression graphs. To respond to what happens (to the garden's population) when the simulation parameters are changed, we explored the outcome of running the model with 10 (120 months) and 40 (480

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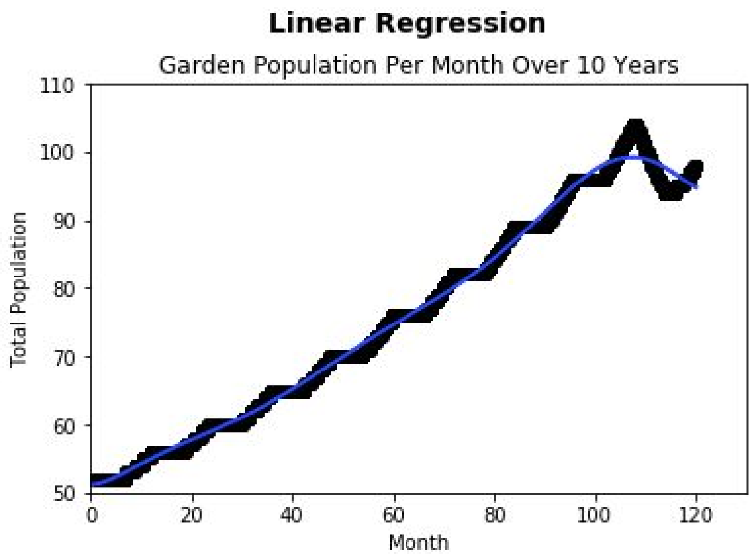
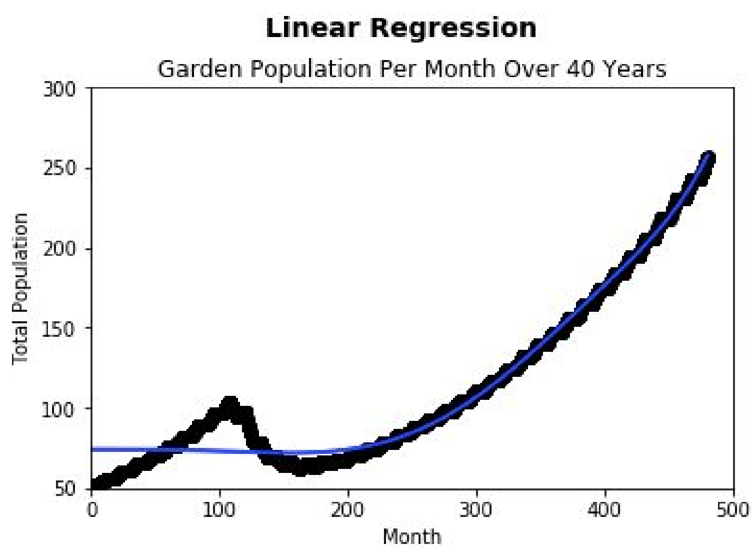
months) years as the overall length of time, or duration of the simulation (2a). The resulting linear regression graphs are shown below:





The coded model and graph both show that the garden's population after 10 years would be about 98 plants and about 257 plants after 40 years. It can be noticed by the graph of the 40 year simulation, that letting the garden continue to grow for twice as long, will result in an overall exponential growth of the garden's total population. Moreover doubling the duration of the garden's growth shows that the garden's total population will be almost 3 times as large

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(83×3=249≅257), which is approximately a 200-210% increase from the 20 year population of 83 plants.

The third and final exploratory question (3) is a comparison between the resulting effects on the garden's final population (over 20 years) from changing the initial population of the plants, the amount of shade provided by the trees, and the climate during the entire simulation. To explore the impact of the plants' initial populations on the garden's total population after 20 years, we increased each plant's initial population (not including the trees) by the same amount and decreased them similarly (3a). The results can be represented by the Google sheets tables below:

Initial Population Increase:

Increase by	Total Population	Deviation
0	83	0
5	115	32+
10	147	64+
20	208	125+
50	384	301+

Initial Population Decrease:

Decrease by	Total Population	Deviation
0	83	0
1	77	6-
2	71	12-
5	51	32-
all but 1	6	77-

These tables show that the garden's total population after 20 years will be affected by the initial population of the four plants. Noticeably, increasing the initial population of each plant by a small amount such as 5, will increase the final garden's population by about 32 plants (≅ 39%), and increasing them by a much larger amount such as 50, will increase the garden's population by about 301 plants (≅ 363%). Opposingly, decreasing the initial population of each plant by a small amount such as 1, will decrease the final garden's population by about 6 plants (≅ 7%), and decreasing them by enough so that the initial population of each plant is 1, will decrease the garden's population by about 77 plants (≅ 93%). Given the amounts that each plant was increased/decreased by, the largest deviation of the resulting total population from the original

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total population (83 plants) is about 301 plants (≅ 363%). This indicates that there will be a significant change in the garden's total population after 20 years if the initial population of the four plants is significantly increased/decreased.

To explore the impact of the shade present in the garden (from the trees' canopies) on the garden's total population after 20 years, we analyzed precisely two factors that play a role in providing shade; the size of the trees' canopies and the number of trees present (3b). To investigate the former of these factors, the radius rate that corresponds to creating the are covered by the trees' canopies was increased/decreased so that the canopies would then be larger/smaller. The results can be represented by the tables below:

Radius Rate Increase:

Increase by	Radius Rate	Total Population	Deviation
0	0.8	83	0
0.1	0.9	83	0
0.2	1	83	0
0.5	1.3	83	0
1	1.8	83	0
2	2.8	83	0

Radius Rate Decrease:

Decrease by	Radius Rate	Total Population	Deviation
0	0.8	83	0
0.1	0.7	84	1+
0.2	0.6	84	1+
0.4	0.4	221	138+
0.6	0.2	261	178+
0.8	0	261	178+

The tables above show that increasing the radius rate of the trees' canopies will have no impact on the garden's total population after 20 years. This is likely due to the fact that the original radius rate will cause the garden to be completely covered by shade at 20 years (as previously mentioned), and hence, increasing their size will not have any greater effect on the garden's final population. Conversely, decreasing the the radius rate of the trees' canopies by a small amount

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such as 0.1 will have little impact on the garden's total population (increase by about 1 plant ≅ 1%), and decreasing it by a significantly large amount such as 0.6 or removing it entirely will have a significant impact on the garden's total population (increase by 178 plants ≅ 214%). Notice that a radius rate decrease in 0.2 has a total population increase of only 1 plant (≅ 1%), while a radius rate decrease of 0.4 (just a 0.2 difference) has a total population increase of 138 plants (≅ 214%). The large jump in deviations from using a radius rate of 0.6 to using a radius rate of 0.4 cannot be easily explained and would require more in depth analysis. To investigate the latter factor that plays a role in providing shade for the garden, precisely the number of trees present, the number of trees planted was increased/decreased. The results can be represented by the tables below:

Tree Population Increase:

Increase by	Number of Trees	Total Population	Deviation
0	12	83	0
1	13	83	0
2	14	83	0
5	17	83	0
10	22	83	0
20	32	83	0

Tree Population Decrease:

Decrease by	Number of Trees	Total Population	Deviation
0	12	83	0
1	11	84	1+
2	10	84	1+
9	3	221	138+
10	2	261	178+
12	0	261	178+

As shown by the tables above, increasing the number of trees planted will have no impact on the garden's total population after 20 years. Like with increasing the radius rate, this is likely because the original number of trees planted will result in the total coverage of the garden by shade at 20 years, and hence, adding more will not have any greater effect on the garden's final population. Interestingly enough, the effect of decreasing the number of trees on the total

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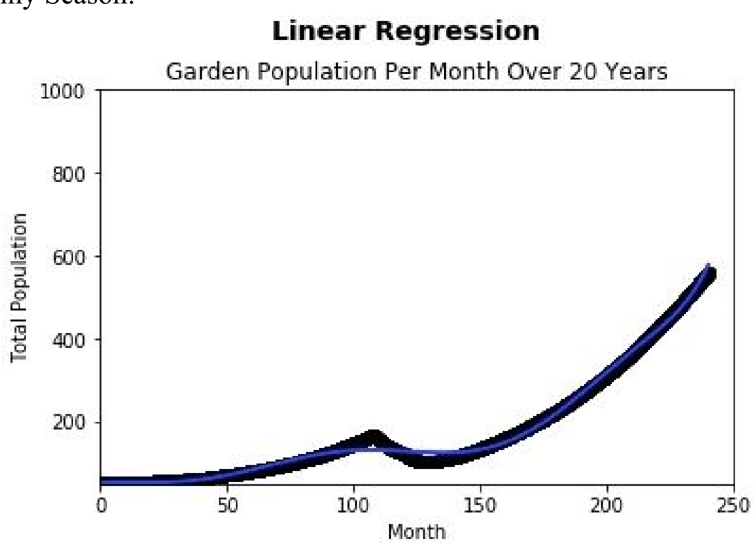
population is proportional to the effect of decreasing the radius rate of the trees' canopies. That is for example, the amount of shade given by 3 trees with radius rates of 0.8 ft/day will be directly proportional (exactly the same) to the amount of shade given by 12 trees with radius rates of 0.4 ft/day, as they both result in the same final garden population of 221 plants (a deviation of 138 from the original). For this reason, the conclusions about the effect of increasing/decreasing the original tree population on the total garden population after 20 years are the same as the ones made about the effect of increasing/decreasing the radius rate of the trees' canopies.
Given the results from changing these two factors, it can be seen that the amount of shade present in the garden will have some effect on the overall population of the garden after 20 years (only when decreasing the radius rate of the trees' canopies or the number of trees planted). Moreover, the largest deviation of the resulting total population from the original total population (83 plants) with respect to shade present in the garden is about 178 plants (≅ 214%).

To explore the third and final part of the third question which is concerned with the impact of climate on the garden's total population, we observed the resulting garden populations from having only a rainy season and only a dry season (3c). Precisely, two linear regression graphs show this affect:

Rainy Season:



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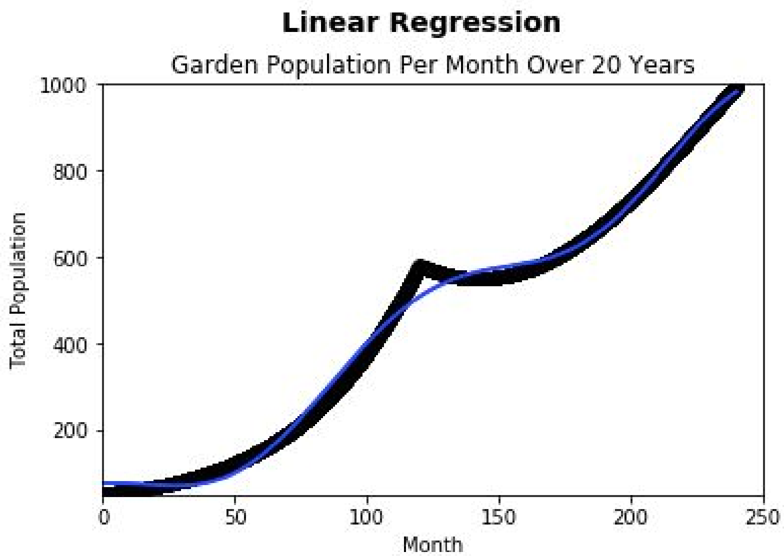
Dry Season:



The information given by the alterations to the model and represented by the graphs above show that the population if there were only a rainy season, would be roughly 557 plants (deviation of 474) and 995 plants if there were only a dry season (deviation of 912). The reason for this is specifically because of the slightly higher growth rate of the sedums in the dry season (0.3). If the sedums had only a growth rate that was 0.1 lower (0.2), then the effects of the rainy and dry seasons on the garden's total population would be the same as each season would have two plants with growth rates of -0.1 and two plants with growth rates of 0.2. Either way, a change in the climate of the simulation, that is, having only a rainy or dry season for the entirety of the 20 year period, would have a significantly large impact on the garden's final population. Precisely, the gardens total population after 20 years with only a dry season, would have a deviation of about 912 plants (≅ 1,099%) from the original (83 plants).

Lastly, to compare the three parts of the third question, that is, the resulting effects on the garden's final population (over 20 years) from changing the initial population of the plants, the amount of shade provided by the trees, and the climate during the entire simulation, we consider the largest deviations of the resulting total population from the original total population (83 plants). The largest deviation in population with respect to the change in the initial populations of the plants (not including trees) was shown to be about 301 plants (≅ 363%), the largest deviation in population with respect to the change in the amount of shade given by the trees was shown to be about 178 plants (≅ 214%), and the largest deviation in population with respect to the change in the climate was shown to be about 912 plants (≅ 1,099%). Therefore, the relative impact of climate on the garden's total population after 20 years is largest.

Conclusion:

This project allowed for the study of a simulation of the overall population growth of a garden (four types of plants) over 20 years given the previously mentioned constraints. Using the logistic equation for constrained growth in this model, allowed for the prediction of the final population of the garden after 20 years, which was shown to be about 83 plants. Moreover, Python was used as a platform for this simulation and for the creation of the linear regression graphs that formed a large part of the project's analysis. As demonstrated above, we were able to conclude that an extended duration of the simulation would result in an exponential growth of the garden's total population and that the largest impact on this population would be caused by a change in climate. The former of the two may be a result of error as the population growing without bound and showing an exponential behavior is not in accordance with the behavior of a standard logistic model. Moreover, we struggled with the calculation of time in our code. We initially didn't have the correct code to calculate the number of simulations to run through the sequence. We had initially used two **for** loops, but we were advised that in doing so, our calculations would be incorrect. We also struggled with how to determine whether it was the dry season or the rainy season based on months. We managed to work through these problems (with Professor Wiseman's help) and give the answer that we found. We were also concerned by the fact that as the number of years increased, the graphs appeared to show exponential growth instead of logistic growth. The idea of logistic growth is that the population grows exponentially and then plateaus as it reaches a certain population. As mentioned however, our graphs don't demonstrate the constraint of 1 plant per sq ft even though we used the logistic equation to calculate the population. This might suggest that we are missing a certain element in our calculations of the population. A way to improve the code would be to include functions (growth function, canopy function, season function) and call them in the **for** loop. This would make changing the constraints easier and it would decrease the runtime.

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